

Hardy inequality and the construction of infinitesimal operators with non-basis family of eigenvectors

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Abstract

We introduce some special Hilbert spaces to present a number of infinitesimal operators with complete minimal non-basis family of eigenvectors. The famous Hardy inequality plays an important role in this construction. Our construction is closely connected with the recent results of G.Q. Xu, S.P. Yung [1], 2005, and H. Zwart [2], 2010, on Riesz basis property of eigenvectors (eigenspaces) of infinitesimal operators. We also give a generalization of our results to the case of operators on some Banach spaces.

Keywords: Hardy inequality, infinitesimal operator, C_0 -group, Riesz basis, eigenvectors, symmetric basis, right shift operator associated to the basis, \mathcal{H}^∞ -calculus

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1. Introduction

The Hardy inequality (in both its discrete and continuous forms) was discovered at the beginning of XX century and has a lot of applications in various branches of mathematics such as analysis, differential equations, mathematical physics, differential geometry and others [3, 4, 5]. The discrete form of it reads [6] that if $p > 1$ and $\{a_k\}_{k=1}^\infty$ is a sequence of nonnegative real numbers, then

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=1}^n a_k \right)^p \leq \left(\frac{p}{p-1} \right)^p \sum_{n=1}^{\infty} a_n^p. \quad (1)$$

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Note that G.H. Hardy came to the discovery of inequality (1) when he tried to obtain an elementary proof of (the weak form of) the Hilbert inequality [7, 3]. The latter inequality was discovered when D. Hilbert studied the solutions to a certain integral equations, see [8], and the weak form of it asserts that if $\{a_n\}_{n=1}^\infty, \{b_n\}_{n=1}^\infty \in \ell_2$ and $a_n \geq 0, b_n \geq 0$, then the double series $\sum_{n=1}^\infty \sum_{m=1}^\infty \frac{a_m b_n}{m+n}$ converges. We must also mark an essential contribution of such mathematicians as M. Riesz, E. Landau and I. Schur to the development of (1), for more details see, e.g. [3].

In what follows by H we denote a separable Hilbert space with norm $\|\cdot\|$ and scalar product $\langle \cdot, \cdot \rangle$. A remarkable result in the spectral theory of C_0 -semigroups in Hilbert spaces was obtained in [1, 2], where the Riesz basis property for eigenvectors of certain class of generators of C_0 -groups was established. We give here a formulation of the main result of these works in some special case.

Theorem 1 ([2]). *Let A be the generator of the C_0 -group on H with eigenvalues $\{\lambda_n\}_{n=1}^\infty$ (counting with multiplicity) and the corresponding (normalized) eigenvectors $\{e_n\}_{n=1}^\infty$. If the following two conditions hold,*

1. $\overline{\text{Lin}}\{e_n\}_{n=1}^\infty = H$;
2. *The point spectrum has a uniform gap, i.e.,*

$$\inf_{n \neq m} |\lambda_n - \lambda_m| > 0, \quad (2)$$

then $\{e_n\}_{n=1}^\infty$ forms a Riesz basis of H .

We note that Theorem 1 follows from the main result of [1], but the approach used by H. Zwart in [2] differs greatly from one proposed by G.Q. Xu and S.P. Yung in [1]. The proof of Theorem 1 in [2] is based, on the one hand, on Carleson's interpolation theorem [9], and, on the other hand, on the fact that each generator of a C_0 -group on H has a bounded \mathcal{H}^∞ -calculus on a strip [10, 11, 12].

Let the eigenvalues $\{\lambda_n\}_{n=1}^\infty$ of the generator A of the C_0 -group on H can be grouped into K sets $\{\lambda_{n,1}\}_{n=1}^\infty, \{\lambda_{n,2}\}_{n=1}^\infty, \dots, \{\lambda_{n,K}\}_{n=1}^\infty$ with $\inf_{n \neq m} |\lambda_{n,k} - \lambda_{m,k}| > 0, k = 1, \dots, K$, and the span of the generalized eigenvectors of A is dense. Then, as it is shown by H. Zwart in [2], there exists a sequence of spectral projections $\{P_n\}_{n=1}^\infty$ of A such that $\{P_n H\}_{n=1}^\infty$ forms a Riesz basis of subspaces in H with $\max_n \dim P_n H = K$. More about Riesz bases of subspaces one can find, e.g., in [13].

The main goal of our work is to show that the assumption (2) in Theorem 1 is obligatory, i.e. if we refuse this assumption or only weaken it, the statement of the Theorem 1 becomes false. The case when $\{\lambda_n\}_{n=1}^\infty$ can be decomposed into K sets, with every set satisfying (2), was affected in [2]. That's why in Theorem 8 we consider the case when $\{\lambda_n\}_{n=1}^\infty$ do not satisfy (2) and, moreover, cannot be decomposed into K sets, with every set satisfying (2), and present the construction of the generator of the C_0 -group with eigenvalues $\{\lambda_n\}_{n=1}^\infty$ and complete minimal non-basis family of eigenvectors. It must be emphasized that the famous Hardy inequality (1) (for $p = 2$) plays a crucial role in the proof of Theorem 8.

Furthermore, we expand this result in Theorem 12 and present the class of infinitesimal operators with complete minimal non-basis family of eigenvectors. Thereby, we demonstrate that a Theorem 1 cannot be improved. For these purposes, we introduce function classes \mathcal{S}_k , $k \in \mathbb{N}$, and present classes of Hilbert spaces $H_k(\{e_n\})$, $k \in \mathbb{N}$, depending on H and on chosen Riesz basis $\{e_n\}_{n=1}^\infty$ of H , and prove that $\{e_n\}_{n=1}^\infty$ do not form a (Schauder) basis of $H_k(\{e_n\})$. Along with this, we indicate some properties of $H_k(\{e_n\})$ and study the properties of the sequence $\{(I - T)^k e_n\}_{n=1}^\infty$, where $Te_n = e_{n+1}$, $n \in \mathbb{N}$, in H .

Further, we propose certain development of the question above. Namely, we use the same idea for the construction of infinitesimal operators, acting on certain Banach sequence spaces, with complete minimal non-basis family of eigenvectors. This construction is essentially based on the Hardy inequality (1) for $p > 1$ and is similar to the one given by Theorem 8.

Also, we generalize this result in Theorem 16 and present the class of infinitesimal operators on Banach spaces with complete minimal non-basis family of eigenvectors. To this end, we use function classes \mathcal{S}_k , $k \in \mathbb{N}$, introduce certain classes of Banach sequence spaces $\ell_{p,k}(\{e_n\})$, $p \geq 1$, $k \in \mathbb{N}$, depending on given ℓ_p space and on arbitrary chosen symmetric basis $\{e_n\}_{n=1}^\infty$ of ℓ_p , and show that $\{e_n\}_{n=1}^\infty$ do not form a (Schauder) basis of $\ell_{p,k}(\{e_n\})$. The concept of symmetric basis was first introduced and studied by I. Singer [14] in connection with the one S. Banach's problem from isomorphic theory of Banach spaces. For various properties of symmetric bases see, e.g., [15, 16]. Finally, we note that the properties of spaces $\ell_{p,k}(\{e_n\})$ and the properties of the sequence $\{(I - T)^k e_n\}_{n=1}^\infty$ in ℓ_p are analogous to the properties of $H_k(\{e_n\})$ and $\{(I - T)^k e_n\}_{n=1}^\infty$ in H , respectively.

2. Auxiliary constructions and preliminary results

2.1. Spaces $H_k(\{e_n\})$, $k \in \mathbb{N}$

We begin by introducing the following definition.

Definition 2. Let E be a Banach space with basis $\{e_n\}_{n=1}^\infty$. Then the operator T defined on E by $Te_n = e_{n+1}$, $n \in \mathbb{N}$, will be called by the right shift operator associated to the basis $\{e_n\}_{n=1}^\infty$.

Suppose that $\{e_n\}_{n=1}^\infty$ is an arbitrary Riesz basis of H and T is the right shift operator associated to $\{e_n\}_{n=1}^\infty$. We introduce the following spaces,

$$H_k^0(\{e_n\}) = \left\{ x \in H : \|x\|_k = \|(I - T)^k x\| \right\}, \quad k \in \mathbb{N}.$$

Note that $H_k^0(\{e_n\})$ is a normed linear space, but not complete, since $0 \in \sigma((I - T)^k)$ for any k . By $H_k(\{e_n\})$ we will call the completion of $H_k^0(\{e_n\})$ in the norm $\|\cdot\|_k$. Using the characteristic property of Riesz basis we have that

$$H = \left\{ x = \sum_{n=1}^\infty c_n e_n : \{c_n\}_{n=1}^\infty \in \ell_2 \right\}.$$

Further we observe that

$$\begin{aligned} \|x\|_k &= \left\| (I - T)^k \sum_{n=1}^\infty c_n e_n \right\| \\ &= \left\| \sum_{n=1}^\infty c_n (e_n - C_k^1 e_{n+1} + \cdots + (-1)^{k-1} C_k^{k-1} e_{n+k-1} + (-1)^k e_{n+k}) \right\| \\ &= \left\| \sum_{n=1}^\infty (c_n - C_k^1 c_{n-1} + \cdots + (-1)^{k+1} C_k^{k-1} c_{n-k+1} + (-1)^k c_{n-k}) e_n \right\|, \end{aligned}$$

where we set $c_{1-j} = 0$, $j \in \mathbb{N}$. The last norm is finite if and only if the condition

$$\sum_{n=1}^\infty |c_n - C_k^1 c_{n-1} + \cdots + (-1)^k c_{n-k}|^2 < \infty$$

holds. Consequently, for each $k \in \mathbb{N}$, $H_k(\{e_n\})$ is a space of formal series $x = (\mathbf{f}) \sum_{n=1}^\infty c_n e_n$ with the property

$$\{c_n - C_k^1 c_{n-1} + \cdots + (-1)^k c_{n-k}\}_{n=1}^\infty \in \ell_2.$$

It follows that $H_k(\{e_n\})$ is a Hilbert space with norm

$$\|x\|_k = \left\| \left(\text{f} \right) \sum_{n=1}^{\infty} c_n e_n \right\|_k = \left\| \sum_{n=1}^{\infty} (c_n - C_k^1 c_{n-1} + \cdots + (-1)^k c_{n-k}) e_n \right\|, \quad (3)$$

$x \in H_k(\{e_n\})$, and a scalar product

$$\langle x, y \rangle_k = \left\langle (I - T)^k x, (I - T)^k y \right\rangle, \quad x, y \in H_k(\{e_n\}).$$

E.g., for any $\alpha \in [0, \frac{1}{2})$, $(\text{f}) \sum_{n=1}^{\infty} n^\alpha e_n \in H_1(\{e_n\})$. Indeed, for large n we have that

$$n^\alpha - (n-1)^\alpha \sim n^{\alpha-1}$$

and, hence, $\{n^\alpha - (n-1)^\alpha\}_{n=1}^{\infty} \in \ell_2$. Concerning the inner product $\langle \cdot, \cdot \rangle_k$ we can say a little more. If $\{e_n\}_{n=1}^{\infty}$ is an orthonormal basis of H , $x = (\text{f}) \sum_{n=1}^{\infty} c_n e_n \in H_k(\{e_n\})$, $y = (\text{f}) \sum_{n=1}^{\infty} d_n e_n \in H_k(\{e_n\})$, then

$$\langle x, y \rangle_k = \sum_{n=1}^{\infty} (c_n - C_k^1 c_{n-1} + \cdots + (-1)^k c_{n-k}) (\bar{d}_n - C_k^1 \bar{d}_{n-1} + \cdots + (-1)^k \bar{d}_{n-k}).$$

Note that, in particular case when $H = \ell_2$ and $\{e_n\}_{n=1}^{\infty}$ denotes the canonical basis of ℓ_2 , $H_k(\{e_n\}) = \ell_2(\Delta^k)$. A sequence space $\ell_2(\Delta^k)$ is the space consisting of all sequences whose k^{th} order differences are 2-absolutely summable, with norm $\|x\|_{\ell_2(\Delta^k)} = \|\Delta^k x\|_{\ell_2}$, where Δ is a difference operator, i.e.

$$\Delta = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ -1 & 1 & 0 & 0 & \cdots \\ 0 & -1 & 1 & 0 & \cdots \\ 0 & 0 & -1 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

see [17, 18] and earlier paper [19], where only the case $k = 1$ is considered. In other words, $\ell_2(\Delta^k) = \{x = \{\alpha_n\}_{n=1}^{\infty} : \Delta^k x \in \ell_2\}$. Hence,

$$H_k(\{e_n\}) = \left\{ x = (\text{f}) \sum_{n=1}^{\infty} c_n e_n : \{c_n\}_{n=1}^{\infty} \in \ell_2(\Delta^k) \right\}, \quad k \in \mathbb{N}, \quad (4)$$

Thus, our class of spaces $H_k(\{e_n\})$ are analogous to $\ell_2(\Delta^k)$, which were studied in [17, 19, 18]. Moreover, we note that $\ell_2(\Delta^k)$ naturally arises as a completion of $(\ell_2)_k^0(\{e_n\})$, where $\{e_n\}_{n=1}^{\infty}$ is a canonical basis of ℓ_2 . The following proposition indicates some properties of the space $H_k(\{e_n\})$.

Proposition 3. *The space $H_k(\{e_n\})$ has the following properties.*

1. $\overline{\text{Lin}}\{e_n\}_{n=1}^\infty = H_k(\{e_n\})$;
2. $\{e_n\}_{n=1}^\infty$ does not form a basis of $H_k(\{e_n\})$;
3. $\{e_n\}_{n=1}^\infty$ has a unique biorthogonal system

$$\left\{ \chi_n = (I - T)^{-k} (I - T^*)^{-k} e_n^* \right\}_{n=1}^\infty$$

in $H_k(\{e_n\})$, where $\langle e_n, e_m^* \rangle = \delta_n^m$;

4. $\{\chi_n\}_{n=1}^\infty$ is uniformly minimal sequence in $H_k(\{e_n\})$ while $\{e_n\}_{n=1}^\infty$ is minimal but not uniformly minimal sequence in $H_k(\{e_n\})$;
5. $H \subset H_1(\{e_n\}) \subset H_2(\{e_n\}) \subset H_3(\{e_n\}) \subset \dots$;
6. $H_k(\{e_n\})$ is a separable Hilbert space, isomorphic to ℓ_2 ;
7. $H_k(\{e_n\})$ has an orthonormal basis;
8. $L = \left\{ x = (\text{f}) \sum_{n=1}^\infty c_n e_n \in H_k(\{e_n\}) : \{c_n\}_{n=1}^\infty \in \ell_2(\Delta^k) \cap c_0 \right\}$, where c_0 is the space of sequences $\{\alpha_n\}_{n=1}^\infty$ with $\lim_{n \rightarrow \infty} \alpha_n = 0$, is not a subspace of $H_k(\{e_n\})$.

PROOF. 1. It follows from the fact that only zero is orthogonal to all e_n , $n \in \mathbb{N}$, with respect to scalar product $\langle \cdot, \cdot \rangle_k$.

2. This is a consequence of the fact that $(\text{f}) \sum_{n=1}^\infty e_n \in H_k(\{e_n\})$ can not be represented by the convergent in $H_k(\{e_n\})$ series $\sum_{n=1}^\infty c_n e_n$, since $\inf_n \|e_n\|_k > 0$.

3. Clearly, $\langle e_n, \chi_j \rangle_k = \left\langle (I - T)^k e_n, (I - T)^k (I - T)^{-k} (I - T^*)^{-k} e_j^* \right\rangle = \delta_n^j$, and the uniqueness of $\{\chi_n\}_{n=1}^\infty$ follows from 1.

4. It is true since $\sup_n \|e_n\|_k < \infty$ while $\sup_n \|\chi_n\|_k = \infty$.

5. It follows from the chain of inclusions $\ell_2 \subset \ell_2(\Delta) \subset \ell_2(\Delta^2) \subset \ell_2(\Delta^3) \subset \dots$ [18].

6. Fix $x = (\text{f}) \sum_{n=1}^\infty c_n e_n \in H_k(\{e_n\})$ and denote $(\Delta^k c)_n = c_n - C_k^1 c_{n-1} + \dots + (-1)^k c_{n-k}$.

Combining (3) with the property of Riesz basis in H , we obtain the following inequality,

$$m \sum_{n=1}^\infty |(\Delta^k c)_n|^2 \leq \|x\|_k^2 \leq M \sum_{n=1}^\infty |(\Delta^k c)_n|^2,$$

which generates an isomorphism between $H_k(\{e_n\})$ and $\ell_2(\Delta^k)$. And, since $\ell_2(\Delta^k)$ is isometrically isomorphic to ℓ_2 [18], $H_k(\{e_n\})$ is isomorphic to ℓ_2 . Hence, $H_k(\{e_n\})$ is a separable space.

7. This is a consequence of the well-known fact that every separable Hilbert space has an orthonormal basis.

8. The proof is based on the fact that $\ell_2(\Delta^k) \cap c_0$ is not closed in $\ell_2(\Delta^k)$. \square

For example, if $\{e_n\}_{n=1}^\infty$ is an orthonormal basis of H , then it is clear that the sequence

$$\left\{ (I - T)^{-k} e_n \right\}_{n=1}^\infty$$

forms an orthonormal basis of $H_k(\{e_n\})$. From the other hand, it is interesting to construct a non-Riesz basis of $H_k(\{e_n\})$. We recall that the first example of non-Riesz basis appeared only in 1948 and it was given by K.I. Babenko in [20]. He showed that for every $\alpha \in (-\frac{1}{2}, 0) \cup (0, \frac{1}{2})$, system of functions $\{|t|^\alpha e^{int}\}_{n=-\infty}^\infty$ forms non-Riesz basis in $L_2(-\pi, \pi)$. This example was later generalized by V.F. Gaposhkin [21] and operators generating non-Riesz bases in H were studied by A.M. Olevskii in [22].

2.2. Spaces $\ell_{p,k}(\{e_n\})$, $p \geq 1$, $k \in \mathbb{N}$

Similarly to the above we introduce the space $\ell_{p,k}(\{e_n\})$ as a completion of the space

$$\ell_{p,k}^0(\{e_n\}) = \left\{ x \in \ell_p : \|x\|_k = \left\| (I - T)^k x \right\| \right\}, \quad k \in \mathbb{N},$$

where $\{e_n\}_{n=1}^\infty$ is a symmetric basis of ℓ_p , $p \geq 1$, and T is the right shift operator associated to $\{e_n\}_{n=1}^\infty$. It is known that the spaces ℓ_p , $p \geq 1$, have a unique, up to equivalence, symmetric basis [15] and it is equivalent to the canonical basis. Thus we arrive at the following assertion.

Proposition 4. *Let $\{e_n\}_{n=1}^\infty$ be a symmetric basis of ℓ_p , $p \geq 1$. Then there exist constants $M \geq m > 0$ such that for each $x = \sum_{n=1}^\infty c_n e_n \in \ell_p$,*

$$m \sum_{n=1}^\infty |c_n|^p \leq \|x\|^p \leq M \sum_{n=1}^\infty |c_n|^p.$$

Proposition 4 says that the class of Riesz bases in ℓ_2 coincide with the class of symmetric bases.

Using Proposition 4 we obtain that

$$\ell_p = \left\{ x = \sum_{n=1}^\infty c_n e_n : \{c_n\}_{n=1}^\infty \in \ell_p \right\}$$

and, consequently, for each $k \in \mathbb{N}$,

$$\ell_{p,k}(\{e_n\}) = \left\{ x = (\text{f}) \sum_{n=1}^{\infty} c_n e_n : \{c_n - C_k^1 c_{n-1} + \cdots + (-1)^k c_{n-k}\}_{n=1}^{\infty} \in \ell_p \right\}$$

is a Banach sequence space with norm

$$\|x\|_k = \left\| (\text{f}) \sum_{n=1}^{\infty} c_n e_n \right\|_k = \left\| \sum_{n=1}^{\infty} (c_n - C_k^1 c_{n-1} + \cdots + (-1)^k c_{n-k}) e_n \right\|, \quad x \in \ell_{p,k}(\{e_n\}).$$

Except the case $p = 2$, the space $\ell_{p,k}(\{e_n\})$, $k \in \mathbb{N}$, is not an inner product space and, hence, not a Hilbert space. On the other hand, $\ell_{2,k}(\{e_n\})$, $k \in \mathbb{N}$, is a Hilbert space since $\ell_{2,k}(\{e_n\}) = H_k(\{e_n\})$, where $H = \ell_2$ and $\{e_n\}_{n=1}^{\infty}$ is a Riesz basis of ℓ_2 . Also we note that, if $\{e_n\}_{n=1}^{\infty}$ denotes the canonical basis of ℓ_p , then $\ell_{p,k}(\{e_n\}) = \ell_p(\Delta^k)$, where $\ell_p(\Delta^k)$ is the space consisting of all sequences whose k^{th} order differences are p -absolutely summable, with norm $\|x\|_{\ell_p(\Delta^k)} = \|\Delta^k x\|_{\ell_p}$, see [17, 18, 19] for details. In other words, $\ell_p(\Delta^k) = \{x = \{\alpha_n\}_{n=1}^{\infty} : \Delta^k x \in \ell_p\}$. It follows that

$$\ell_{p,k}(\{e_n\}) = \left\{ x = (\text{f}) \sum_{n=1}^{\infty} c_n e_n : \{c_n\}_{n=1}^{\infty} \in \ell_p(\Delta^k) \right\}, \quad k \in \mathbb{N}.$$

In the following assertion we collect some properties of the space $\ell_{p,k}(\{e_n\})$.

Proposition 5. *Let $\{e_n\}_{n=1}^{\infty}$ be a symmetric basis of ℓ_p , $p \geq 1$, and $k \in \mathbb{N}$. Then the following statements hold true.*

1. $\overline{\text{Lin}}\{e_n\}_{n=1}^{\infty} = \ell_{p,k}(\{e_n\})$;
2. $\{e_n\}_{n=1}^{\infty}$ does not form a basis of $\ell_{p,k}(\{e_n\})$;
3. If $p > 1$, then $\{e_n\}_{n=1}^{\infty}$ has a unique biorthogonal system

$$\left\{ \chi_n = (I - T)^{-k} (I - T^*)^{-k} e_n^* \right\}_{n=1}^{\infty}$$

in $(\ell_{p,k}(\{e_n\}))^*$, where $\{e_n^*\}_{n=1}^{\infty}$ is biorthogonal to $\{e_n\}_{n=1}^{\infty}$ basis of ℓ_q , where $\frac{1}{p} + \frac{1}{q} = 1$;

4. If $p > 1$, then $\{\chi_n\}_{n=1}^{\infty}$ is uniformly minimal sequence in $(\ell_{p,k}(\{e_n\}))^*$ while $\{e_n\}_{n=1}^{\infty}$ is minimal but not uniformly minimal sequence in $\ell_{p,k}(\{e_n\})$;
5. $\ell_p \subset \ell_{p,1}(\{e_n\}) \subset \ell_{p,2}(\{e_n\}) \subset \ell_{p,3}(\{e_n\}) \subset \cdots$;
6. $\ell_{p,k}(\{e_n\})$ is a separable Banach sequence space, isomorphic to ℓ_p ;
7. $L = \left\{ x = (\text{f}) \sum_{n=1}^{\infty} c_n e_n \in \ell_{p,k}(\{e_n\}) : \{c_n\}_{n=1}^{\infty} \in \ell_p(\Delta^k) \cap c_0 \right\}$ is not a subspace of $\ell_{p,k}(\{e_n\})$.

The proof of Proposition 5 goes similarly to the proof of Proposition 3.

2.3. The sequence $\{(I - T)^k e_n\}_{n=1}^\infty$, $k \in \mathbb{N}$

The next proposition involves the concept of Riesz basis and the notion of the right shift operator to the construction of some dense in H systems $\{\phi_n^k\}_{n=1}^\infty$, which have a unique biorthogonal systems, but each infinite subsequence $\{\phi_n^k\}_{n \in \mathcal{G}}$, $\mathcal{G} \subseteq \mathbb{N}$, $|\mathcal{G}| = \infty$, do not form a basis of any subspace $L \subseteq H$. In other words it means that each subsequence $\{\phi_n^k\}_{n \in \mathcal{G}}$, $\mathcal{G} \subseteq \mathbb{N}$, $|\mathcal{G}| = \infty$, is not a basis sequence.

Proposition 6. *Let $\{e_n\}_{n=1}^\infty$ be a Riesz basis of H with biorthogonal basis $\{e_n^*\}_{n=1}^\infty$, T is the right shift operator associated to the basis $\{e_n\}_{n=1}^\infty$ and $k \in \mathbb{N}$. Then for the sequence $\{\phi_n^k\}_{n=1}^\infty$, where $\phi_n^k = (I - T)^k e_n$, $n \in \mathbb{N}$, we have the following.*

1. $\overline{\text{Lin}}\{\phi_n^k\}_{n=1}^\infty = H$;
2. $\{\phi_n^k\}_{n=1}^\infty$ has a unique complete biorthogonal sequence

$$\left\{ \psi_n^k = (I - T^*)^{-k} e_n^* = \sum_{j_1=1}^n \sum_{j_2=1}^{j_1} \cdots \sum_{j_k=1}^{j_{k-1}} e_{j_k}^* \right\}_{n=1}^\infty.$$

3. $\{\phi_n^k\}_{n=1}^\infty$ does not form a basis of H .
4. For each $\mathcal{G} \subset \mathbb{N}$ with $|\mathcal{G}| = \infty$ the sequence $\{\phi_n^k\}_{n \in \mathcal{G}}$ is not a basis sequence.

PROOF. 1. Assume the opposite, i.e. that there exists $x \in H \setminus \{0\}$ such that $\langle x, \phi_n^k \rangle = 0$, $n \in \mathbb{N}$. It follows that $\langle ((I - T)^k)^* x, e_n \rangle = 0$, $n \in \mathbb{N}$, and $((I - T)^k)^* x = 0$. Since

$$\ker((I - T)^k)^* = \ker(I - T)^* = \{0\},$$

we obtain that $x = 0$, which contradicts to the assumption.

2. The trivial computation gives $\langle \psi_n^k, \phi_j^k \rangle = \delta_n^j$. To prove that $\overline{\text{Lin}}\{\psi_n^k\}_{n=1}^\infty = H$ we assume the opposite, i.e. that there exists $x \in H \setminus \{0\}$ such that $\langle x, \psi_n^k \rangle = 0$, $n \in \mathbb{N}$. Hence $\sum_{j_1=1}^n \sum_{j_2=1}^{j_1} \cdots \sum_{j_k=1}^{j_{k-1}} \langle x, e_{j_k}^* \rangle = 0$, $n \in \mathbb{N}$. This yields $\langle x, e_n^* \rangle = 0$, $n \in \mathbb{N}$, and, therefore, $x = 0$, which contradicts to the assumption. The uniqueness of biorthogonal sequence $\{\psi_n^k\}_{n=1}^\infty$ follows from the statement 1.

3. Assume the opposite, i.e. that $\{\phi_n^k\}_{n=1}^\infty$ is a basis of H . Then $\{\psi_n^k\}_{n=1}^\infty$ also forms a basis of H [13]. Since $\{e_n\}_{n=1}^\infty$ is a Riesz basis of H , there exists a bounded linear invertible

operator $S : H \rightarrow H$ with bounded inverse, such that $e_n = Se'_n$, $n \in \mathbb{N}$, where $\{e'_n\}_{n=1}^\infty$ is an orthonormal basis of H . Consequently, for each $n \in \mathbb{N}$,

$$\begin{aligned}\|\phi_n^k\|^2 &= \|e_n - C_k^1 e_{n+1} + \cdots + (-1)^{k-1} C_k^{k-1} e_{n+k-1} + (-1)^k e_{n+k}\|^2 \\ &= \|S(e'_n - C_k^1 e'_{n+1} + \cdots + (-1)^{k-1} C_k^{k-1} e'_{n+k-1} + (-1)^k e'_{n+k})\|^2 \\ &\geq \frac{1}{\|S^{-1}\|^2} \|e'_n - C_k^1 e'_{n+1} + \cdots + (-1)^{k-1} C_k^{k-1} e'_{n+k-1} + (-1)^k e'_{n+k}\|^2 \\ &= \|S^{-1}\|^{-2} \sum_{j=0}^k (C_k^j)^2 = C_k > 0.\end{aligned}$$

It follows that $\inf_n \|\phi_n^k\| \geq \sqrt{C_k} > 0$. Hence $\sup_n \|\psi_n^k\| \leq \widetilde{C}_k < \infty$ [13].

Now pick $x = \sum_{n=1}^\infty \frac{e_n}{n} \in H$, i.e. $x = \sum_{n=1}^\infty \langle x, e_n^* \rangle e_n$, where $\langle x, e_n^* \rangle = \frac{1}{n}$, $n \in \mathbb{N}$. Since $\{\phi_n^k\}_{n=1}^\infty$ is a basis and $\{\psi_n^k\}_{n=1}^\infty$ is a biorthogonal basis, x has a unique expansion $x = \sum_{n=1}^\infty \langle x, \psi_n^k \rangle \phi_n^k$. Then, from the one hand,

$$\langle x, \psi_n^k \rangle = \sum_{j_1=1}^n \sum_{j_2=1}^{j_1} \cdots \sum_{j_k=1}^{j_{k-1}} \langle x, e_{j_k}^* \rangle = \sum_{j_1=1}^n \sum_{j_2=1}^{j_1} \cdots \sum_{j_k=1}^{j_{k-1}} \frac{1}{j_k} \rightarrow \infty$$

when n tends to ∞ . And, from the other hand, the Cauchy-Schwartz yields

$$\sup_n |\langle x, \psi_n^k \rangle| \leq \|x\| \sup_n \|\psi_n^k\| \leq \widetilde{C}_k \|x\| < \infty.$$

Hence, we arrive at a contradiction.

4. The proof of this fact is similar to the proof of the statement 3. \square

By the proof of Proposition 6 we have that $\{\psi_n^k\}_{n=1}^\infty$ is uniformly minimal sequence while $\{\phi_n^k\}_{n=1}^\infty$ is minimal but not uniformly minimal sequence in H .

The following proposition reflects an interplay between the concept of symmetric basis and the notion of the right shift operator. Moreover, it provides a general method of construction of certain dense in ℓ_p , $p > 1$, systems $\{\phi_n^k\}_{n=1}^\infty$, which have a unique biorthogonal systems in ℓ_q , $\frac{1}{p} + \frac{1}{q} = 1$, but each subsequence $\{\phi_n^k\}_{n \in \mathcal{G}}$, $\mathcal{G} \subseteq \mathbb{N}$, $|\mathcal{G}| = \infty$, is not a basis sequence of ℓ_p .

Proposition 7. *Let $\{e_n\}_{n=1}^\infty$ be a symmetric basis of ℓ_p , $p > 1$, with biorthogonal basis $\{e_n^*\}_{n=1}^\infty \subset \ell_q$, $\frac{1}{p} + \frac{1}{q} = 1$, T is the right shift operator associated to the basis $\{e_n\}_{n=1}^\infty$ and $k \in \mathbb{N}$. Then for the sequence $\{\phi_n^k\}_{n=1}^\infty$, where $\phi_n^k = (I - T)^k e_n$, $n \in \mathbb{N}$, we have the following.*

1. $\overline{\text{Lin}}\{\phi_n^k\}_{n=1}^\infty = \ell_p$;
2. $\{\phi_n^k\}_{n=1}^\infty$ has a unique, complete in ℓ_q , biorthogonal sequence

$$\left\{ \psi_n^k = (I - T^*)^{-k} e_n^* = \sum_{j_1=1}^n \sum_{j_2=1}^{j_1} \cdots \sum_{j_k=1}^{j_{k-1}} e_{j_k}^* \right\}_{n=1}^\infty.$$

3. For each $\mathcal{G} \subseteq \mathbb{N}$ with $|\mathcal{G}| = \infty$ the sequence $\{\phi_n^k\}_{n \in \mathcal{G}}$ is not a basis sequence of ℓ_p .

The proof of Proposition 7 is similar to the proof of Proposition 6 and again we see that $\{\psi_n^k\}_{n=1}^\infty$ is uniformly minimal sequence in ℓ_q while $\{\phi_n^k\}_{n=1}^\infty$ is minimal but not uniformly minimal sequence in ℓ_p .

3. The construction of infinitesimal operators with non-basis family of eigenvectors on Hilbert spaces

3.1. Infinitesimal operators on $H_1(\{e_n\})$

In the following by $[X]$ we denote the space of all bounded linear operators on a Banach space X . Define the operator $A : H_1(\{e_n\}) \supset D(A) \mapsto H_1(\{e_n\})$ as

$$Ax = A(\text{f}) \sum_{n=1}^\infty c_n e_n = (\text{f}) \sum_{n=1}^\infty \lambda_n c_n e_n, \quad (5)$$

with domain

$$D(A) = \left\{ x = (\text{f}) \sum_{n=1}^\infty c_n e_n \in H_1(\{e_n\}) : \{\lambda_n c_n\}_{n=1}^\infty \in \ell_2(\Delta) \right\}, \quad (6)$$

An example of the generator of unbounded C_0 -group with eigenvectors which do not form a basis is given by the following theorem.

Theorem 8. *Let $\{e_n\}_{n=1}^\infty$ be a Riesz basis of H . Then $\{e_n\}_{n=1}^\infty$ does not form a basis of $H_1(\{e_n\})$ and the operator A defined by (5) with domain (6), where $\lambda_n = i \ln n$, generates a C_0 -group on $H_1(\{e_n\})$.*

PROOF. By Proposition 3 we have that $\{e_n\}_{n=1}^\infty$ does not form a basis of $H_1(\{e_n\})$. Since $Ae_n = i \ln n \cdot e_n$, $n \in \mathbb{N}$, for every $t \in \mathbb{R}$ we have $e^{At}e_n = e^{it \ln n}e_n$, $n \in \mathbb{N}$. Hence, $e^{At}x = e^{At}(\text{f}) \sum_{n=1}^\infty c_n e_n = (\text{f}) \sum_{n=1}^\infty e^{it \ln n} c_n e_n$ for $x \in H_1(\{e_n\})$.

First we show that $e^{At} \in [H_1(\{e_n\})]$ for arbitrary $t \in \mathbb{R}$. To this end we observe that, by (3), for each $x = (\mathbf{f}) \sum_{n=1}^{\infty} c_n e_n \in H_1(\{e_n\})$,

$$\|e^{At}x\|_1 = \left\| (\mathbf{f}) \sum_{n=1}^{\infty} e^{it \ln n} c_n e_n \right\|_1 = \left\| \sum_{n=1}^{\infty} (e^{it \ln n} c_n - e^{it \ln(n-1)} c_{n-1}) e_n \right\|,$$

where we set $c_0 = \ln 0 = 0$. Further,

$$\begin{aligned} \|e^{At}x\|_1 &= \left\| \sum_{n=1}^{\infty} (e^{it \ln n} c_n - e^{it \ln n} c_{n-1} + e^{it \ln n} c_{n-1} - e^{it \ln(n-1)} c_{n-1}) e_n \right\| \\ &\leq \left\| \sum_{n=1}^{\infty} e^{it \ln n} (c_n - c_{n-1}) e_n \right\| + \left\| \sum_{n=2}^{\infty} (e^{it \ln n} - e^{it \ln(n-1)}) c_{n-1} e_n \right\|. \end{aligned}$$

Since for $n \geq 2$,

$$e^{it \ln n} - e^{it \ln(n-1)} = e^{it \ln n} \left(1 - e^{it \ln(1 - \frac{1}{n})} \right),$$

using the asymptotics

$$1 - e^{it \ln(1 - \frac{1}{n})} \sim it \ln \left(1 - \frac{1}{n} \right) \sim -\frac{it}{n}$$

for sufficiently large n , we infer that an estimate $\left\| \sum_{n=2}^{\infty} (e^{it \ln n} - e^{it \ln(n-1)}) c_{n-1} e_n \right\| \leq B \|x\|_1$ takes place if and only if an estimate $\left\| \sum_{n=2}^{\infty} e^{it \ln n} c_{n-1} \frac{it}{n} e_n \right\| \leq \tilde{B} \|x\|_1$ holds. Since $\{e_n\}_{n=1}^{\infty}$ is a Riesz basis of H , by virtue of Hardy inequality (1) for $p = 2$, we obtain that

$$\begin{aligned} \left\| \sum_{n=2}^{\infty} e^{it \ln n} c_{n-1} \frac{it}{n} e_n \right\|^2 &= |t|^2 \left\| \sum_{n=2}^{\infty} \left(\sum_{j=1}^{n-1} (c_j - c_{j-1}) \frac{e^{it \ln n}}{n} \right) e_n \right\|^2 \\ &\leq M |t|^2 \sum_{n=2}^{\infty} \left| \sum_{j=1}^{n-1} \frac{c_j - c_{j-1}}{n} \right|^2 |e^{it \ln n}|^2 < M |t|^2 \sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{j=1}^n |c_j - c_{j-1}| \right)^2 \\ &\leq 4M |t|^2 \sum_{n=1}^{\infty} |c_n - c_{n-1}|^2 \leq 4 \frac{M}{m} |t|^2 \left\| \sum_{n=1}^{\infty} (c_n - c_{n-1}) e_n \right\|^2 = 4 \frac{M}{m} |t|^2 \|x\|_1^2. \end{aligned}$$

This implies that $e^{At} \in [H_1(\{e_n\})]$ for any $t \in \mathbb{R}$.

Second, we show the strong continuity of e^{At} . To this end we note that for each $x =$

$$(f) \sum_{n=1}^{\infty} c_n e_n \in H_1(\{e_n\}),$$

$$\begin{aligned} \|e^{At}x - x\|_1 &= \left\| \sum_{n=1}^{\infty} ((e^{it \ln n} - 1)c_n - (e^{it \ln(n-1)} - 1)c_{n-1})e_n \right\| \\ &= \left\| \sum_{n=1}^{\infty} ((e^{it \ln n} - 1)(c_n - c_{n-1})e_n + \sum_{n=2}^{\infty} (e^{it \ln n} - e^{it \ln(n-1)})c_{n-1}e_n \right\| \\ &\leq M \left(\sum_{n=1}^{\infty} |e^{it \ln n} - 1|^2 |c_n - c_{n-1}|^2 \right)^{\frac{1}{2}} + \left\| \sum_{n=2}^{\infty} (e^{it \ln n} - e^{it \ln(n-1)})c_{n-1}e_n \right\|. \end{aligned}$$

The first term tends to zero when $t \rightarrow 0$, since $T(t)x = \sum_{n=1}^{\infty} e^{it \ln n} \langle x, e_n^* \rangle e_n$ is a C_0 -group on H , see [23]. Using similar to the above arguments we see that the second term also tends to zero when $t \rightarrow 0$. Consequently, the strong continuity is proved. Finally, since the group property for e^{At} is obvious, the Theorem 8 is proved. \square

Concerning Theorem 8 we note the following. It turns out that, even if we consider the spectrum $\{\lambda_n\}_{n=1}^{\infty}$ of A defined by (5,6) of the same geometric nature, i.e. satisfying

$$\lim_{n \rightarrow \infty} i\lambda_n = -\infty \quad \text{and} \quad \lim_{n \rightarrow \infty} |\lambda_{n+1} - \lambda_n| = 0, \quad (7)$$

then A not necessary generates a C_0 -group on $H_1(\{e_n\})$. To show this we choose $\lambda_n = i\sqrt{n}$ and prove the following.

Proposition 9. *Let $\{e_n\}_{n=1}^{\infty}$ be a Riesz basis of H . Then the operator A defined by (5) with domain (6), where $\lambda_n = i\sqrt{n}$, does not generate a C_0 -group on $H_1(\{e_n\})$.*

PROOF. By the proof of Theorem 8 we claim that $e^{At} \in [H_1(\{e_n\})]$ for any $t \in \mathbb{R}$ if and only if

$$\sum_{n=2}^{\infty} (e^{it\sqrt{n}} - e^{it\sqrt{n-1}})c_{n-1}e_n$$

converges in H for all $x = (f) \sum_{n=1}^{\infty} c_n e_n \in D(A)$ and for any t . Since again, as in the proof of Theorem 8, for $n \geq 2$,

$$e^{it\sqrt{n}} - e^{it\sqrt{n-1}} = e^{it\sqrt{n}} \left(1 - e^{it(\sqrt{n-1}-\sqrt{n})} \right),$$

and using the asymptotics

$$1 - e^{it(\sqrt{n-1}-\sqrt{n})} \sim it(\sqrt{n-1}-\sqrt{n}) \sim -it \frac{1}{\sqrt{n}}$$

for large n , we arrive at the asymptotic series

$$\sum_{n=2}^{\infty} e^{it\sqrt{n}} c_{n-1} \frac{it}{\sqrt{n}} e_n.$$

Further one easily see that the operator

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 & \dots \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 & \dots \\ \frac{1}{\sqrt{4}} & \frac{1}{\sqrt{4}} & \frac{1}{\sqrt{4}} & \frac{1}{\sqrt{4}} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

is unbounded in ℓ_2 . Consequently,

$$\begin{aligned} \left\| \sum_{n=2}^{\infty} e^{it\sqrt{n}} c_{n-1} \frac{it}{\sqrt{n}} e_n \right\|^2 &= |t|^2 \left\| \sum_{n=2}^{\infty} \left(\sum_{j=1}^{n-1} (c_j - c_{j-1}) \frac{e^{it\sqrt{n}}}{\sqrt{n}} \right) e_n \right\|^2 \\ &\geq m|t|^2 \sum_{n=2}^{\infty} \left| \sum_{j=1}^{n-1} \frac{c_j - c_{j-1}}{\sqrt{n}} \right|^2 |e^{it\sqrt{n}}|^2 = m|t|^2 \sum_{n=1}^{\infty} \left| \sum_{j=1}^n \frac{c_j - c_{j-1}}{\sqrt{n+1}} \right|^2 \end{aligned}$$

does not converge for all $x = (\text{f}) \sum_{n=1}^{\infty} c_n e_n \in D(A) \setminus \{0\}$. It follows that there exists $t \neq 0$ such that $e^{At} \notin [H_1(\{e_n\})]$. Hence, the operator A can not generate a C_0 -group on $H_1(\{e_n\})$. \square

The inner reason of the phenomenon described above is as follows. The rotations $e^{it\sqrt{x}}$, are slowing down, when $x \rightarrow +\infty$, with too low speed to guarantee the convergence of

$$\sum_{n=2}^{\infty} (e^{it\sqrt{n}} - e^{it\sqrt{n-1}}) c_{n-1} e_n$$

in H . Using the same arguments as in the proof of Proposition 9, we can say the following.

Proposition 10. *Let $\{e_n\}_{n=1}^{\infty}$ be a Riesz basis of H . Then the operator A defined by (5) with domain (6), where $\{\lambda_n\}_{n=1}^{\infty}$ satisfy (7) and $\lim_{n \rightarrow \infty} \frac{|\lambda_n|}{\sqrt{n}} > 0$, does not generate a C_0 -group on $H_1(\{e_n\})$.*

3.2. Expanding of Theorem 8 and infinitesimal operators on $H_k(\{e_n\})$

We expand the Theorem 8 in two directions. Namely, from the one hand, we extend the class of Hilbert spaces where infinitesimal operators with complete minimal, non-basis family

of eigenvectors act. And, from other hand, we consider more general behaviour of the spectrum of these operators. For this purpose we define the following function classes.

Definition 11. Let $f : [1, +\infty) \mapsto \mathbb{R}$ be a real function and let $k \in \mathbb{N}$. Then we define

$$\mathcal{S}_k = \left\{ f : \lim_{x \rightarrow \infty} f(x) = +\infty; \lim_{n \rightarrow \infty} n^k |f(n-j) - f(n)| < \infty \text{ for any } j = 1, \dots, k \right\}.$$

For example, $f(x) = \ln x \in \mathcal{S}_1$, $g(x) = \ln \ln \sqrt{x+1} \in \mathcal{S}_1$ and, clearly, we have the chain of inclusions $\mathcal{S}_1 \supset \mathcal{S}_2 \supset \mathcal{S}_3 \supset \dots$. Now we formulate our generalization.

Theorem 12. Assume that $\{e_n\}_{n=1}^\infty$ is a Riesz basis of H , $k \in \mathbb{N}$ and let T be the right shift operator associated to $\{e_n\}_{n=1}^\infty$. Then $\{e_n\}_{n=1}^\infty$ does not form a basis of $H_k(\{e_n\})$ and the operator $A_k : H_k(\{e_n\}) \supset D(A_k) \mapsto H_k(\{e_n\})$, defined by

$$A_k x = A_k(\mathfrak{f}) \sum_{n=1}^\infty c_{k,n} e_n = (\mathfrak{f}) \sum_{n=1}^\infty i f_k(n) \cdot c_{k,n} e_n,$$

where $f_k \in \mathcal{S}_k$, with domain

$$D(A_k) = \left\{ x = (\mathfrak{f}) \sum_{n=1}^\infty c_{k,n} e_n \in H_k(\{e_n\}) : \{f_k(n) \cdot c_{k,n}\}_{n=1}^\infty \in \ell_2(\Delta^k) \right\}, \quad (8)$$

generates a C_0 -group on $H_k(\{e_n\})$.

PROOF. Note that Proposition 3 yields that $\{e_n\}_{n=1}^\infty$ does not form a basis of $H_k(\{e_n\})$. Since $A_k e_n = i f_k(n) \cdot e_n$, $n \in \mathbb{N}$, for every $t \in \mathbb{R}$ we have $e^{A_k t} e_n = e^{i t f_k(n)} e_n$, $n \in \mathbb{N}$, and $e^{A_k t} x = e^{A_k t}(\mathfrak{f}) \sum_{n=1}^\infty c_{k,n} e_n = (\mathfrak{f}) \sum_{n=1}^\infty e^{i t f_k(n)} c_{k,n} e_n$ for $x \in H_k(\{e_n\})$.

First we fix $k \in \mathbb{N}$ and show that $e^{A_k t} \in [H_k(\{e_n\})]$ for arbitrary $t \in \mathbb{R}$. To this end we note that for each $x = (\mathfrak{f}) \sum_{n=1}^\infty c_{k,n} e_n \in H_k(\{e_n\})$,

$$\begin{aligned} \|e^{A_k t} x\|_k &= \left\| (\mathfrak{f}) \sum_{n=1}^\infty e^{i t f_k(n)} c_{k,n} e_n \right\|_k \\ &= \left\| \sum_{n=1}^\infty \left(e^{i t f_k(n)} c_{k,n} - C_k^1 e^{i t f_k(n-1)} c_{k,n-1} + \dots + (-1)^k e^{i t f_k(n-k)} c_{k,n-k} \right) e_n \right\| \end{aligned}$$

by (3), where we set $c_{k,1-j} = f_k(1-j) = 0$ for $j \in \mathbb{N}$. Further we claim that

$$\begin{aligned}
\|e^{A_k t} x\|_k &= \left\| \sum_{n=1}^{\infty} \left(e^{itf_k(n)} c_{k,n} - C_k^1 e^{itf_k(n)} c_{k,n-1} + \cdots + (-1)^k e^{itf_k(n)} c_{k,n-k} \right. \right. \\
&\quad \left. \left. + C_k^1 e^{itf_k(n)} c_{k,n-1} - C_k^2 e^{itf_k(n)} c_{k,n-2} + \cdots + (-1)^{k+1} e^{itf_k(n)} c_{k,n-k} \right. \right. \\
&\quad \left. \left. - C_k^1 e^{itf_k(n-1)} c_{k,n-1} + \cdots + (-1)^k e^{itf_k(n-k)} c_{k,n-k} \right) e_n \right\| \\
&= \left\| \sum_{n=1}^{\infty} e^{itf_k(n)} (c_{k,n} - C_k^1 c_{k,n-1} + \cdots + (-1)^k c_{k,n-k}) e_n \right. \\
&\quad \left. + \sum_{n=2}^{\infty} C_k^1 c_{k,n-1} (e^{itf_k(n)} - e^{itf_k(n-1)}) e_n - \sum_{n=3}^{\infty} C_k^2 c_{k,n-2} (e^{itf_k(n)} - e^{itf_k(n-2)}) e_n + \cdots \right. \\
&\quad \left. + (-1)^{k+1} \sum_{n=k+1}^{\infty} c_{k,n-k} (e^{itf_k(n)} - e^{itf_k(n-k)}) e_n \right\|.
\end{aligned}$$

Denote $\beta_{k,j}(n) = n^k(f_k(n-j) - f_k(n))$, $j = 1, \dots, k$. Observe that, since $f_k \in \mathcal{S}_k$, for any $j = 1, \dots, k$, we have that $\lim_{n \rightarrow \infty} \beta_{k,j}(n) < \infty$. Moreover, for any $j = 1, \dots, k$, and each $n \geq j+1$, we have that

$$e^{itf_k(n)} - e^{itf_k(n-j)} = e^{itf_k(n)} \left(1 - e^{it(f_k(n-j) - f_k(n))} \right),$$

and

$$1 - e^{it(f_k(n-j) - f_k(n))} \sim it(f_k(n-j) - f_k(n)) = itn^{-k} \beta_{k,j}(n),$$

for sufficiently large n . Therefore, we infer that for any $j = 1, \dots, k$, an estimate

$$\left\| \sum_{n=j+1}^{\infty} c_{k,n-j} (e^{itf_k(n)} - e^{itf_k(n-j)}) e_n \right\| \leq B_{k,j} \|x\|_k$$

takes place if and only if an estimate $\left\| \sum_{n=j+1}^{\infty} e^{itf_k(n)} c_{k,n-j} \frac{it}{n^k} \beta_{k,j}(n) e_n \right\| \leq \tilde{B}_{k,j} \|x\|_k$ holds.

Since $\{e_n\}_{n=1}^{\infty}$ is a Riesz basis of H , application of the Hardy inequality k times yields for every

$j = 1, \dots, k$, the following.

$$\begin{aligned}
& \left\| \sum_{n=j+1}^{\infty} e^{itf_k(n)} c_{k,n-j} \frac{it}{n^k} \beta_{k,j}(n) e_n \right\|^2 = |t|^2 \left\| \sum_{n=j+1}^{\infty} \left(\sum_{\nu=1}^{n-j} (c_{k,\nu} - c_{k,\nu-1}) \frac{e^{itf_k(n)}}{n^k} \beta_{k,j}(n) \right) e_n \right\|^2 \\
& \leq M|t|^2 \sum_{n=j+1}^{\infty} \left| \sum_{\nu=1}^{n-j} \frac{c_{k,\nu} - c_{k,\nu-1}}{n} \right|^2 \left| \frac{e^{itf_k(n)}}{n^{k-1}} \beta_{k,j}(n) \right|^2 < MD_{k,j} |t|^2 \sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{\nu=1}^n \frac{|c_{k,\nu} - c_{k,\nu-1}|}{n^{k-1}} \right)^2 \\
& \leq 4MD_{k,j} |t|^2 \sum_{n=1}^{\infty} \frac{|c_{k,n} - c_{k,n-1}|^2}{n^{2(k-1)}} = 4MD_{k,j} |t|^2 \sum_{n=1}^{\infty} \left(\frac{1}{n} \left| \sum_{\nu=1}^n (c_{k,\nu} - 2c_{k,\nu-1} + c_{k,\nu-2}) \right| \frac{1}{n^{k-2}} \right)^2 \\
& \leq 4^2 MD_{k,j} |t|^2 \sum_{n=1}^{\infty} \frac{|c_{k,n} - 2c_{k,n-1} + c_{k,n-2}|^2}{n^{2(k-2)}} \leq \dots \\
& \leq 4^k MD_{k,j} |t|^2 \sum_{n=1}^{\infty} \left| c_{k,n} - C_k^1 c_{k,n-1} + C_k^2 c_{k,n-2} + \dots + (-1)^{k+1} C_k^{k-1} c_{k,n-k+1} + (-1)^k c_{k,n-k} \right|^2 \\
& \leq 4^k \frac{M}{m} D_{k,j} |t|^2 \|x\|_k^2.
\end{aligned}$$

It follows that $e^{A_k t} \in [H_k(\{e_n\})]$ for any $t \in \mathbb{R}$. Second, the proof of the strong continuity of $e^{A_k t}$ goes similarly to the proof of the strong continuity of e^{At} in Theorem 8. Finally, since the group property for $e^{A_k t}$ is obvious, the Theorem 12 is proved. \square

Remark 1.

- In 1967 V.E. Katsnel'son [24] proved the following theorem.

Theorem 13. [24] Let $\{\lambda_n\}_{n=1}^{\infty}$ be a sequence of distinct points in the upper half-plane $\{z \in \mathbb{C} : \text{Im}(z) > 0\}$ and assume that

$$\inf_{1 \leq j < \infty} \prod_{k=1; k \neq j}^{\infty} \left| \frac{\lambda_j - \lambda_k}{\lambda_j - \bar{\lambda}_k} \right| = 0. \quad (9)$$

Then there exists a linear operator $A : H \supset D(A) \mapsto H$ such that:

1. $\text{Im}\langle Ax, x \rangle \geq 0$, $x \in D(A)$.
2. The eigenvalues of A are $\{\lambda_n\}_{n=1}^{\infty}$ and $\sigma(A) = \overline{\{\lambda_n\}_{n=1}^{\infty}}$.
3. The system of eigenvectors $\{v_n\}_{n=1}^{\infty}$ of A is dense in H but not uniformly minimal. Furthermore, if $\{\text{Im}(\lambda_n)\}_{n=1}^{\infty}$ is, in addition, a bounded sequence, then there exists a linear operator $A : H \supset D(A) \mapsto H$ satisfying 1-3 and

$$4. A = A_{\Re} + iA_{\Im},$$

where A_{\Re} is selfadjoint operator and A_{\Im} is bounded positive operator.

Note that in any horizontal strip $\{z \in \mathbb{C} : 0 < \text{Im}(z) < \alpha\}$ the condition (9) turns into $\inf_{j \neq k} |\lambda_j - \lambda_k| = 0$, see [9]. It follows that for any sequence $\{\mu_n\}_{n=1}^{\infty}$ of distinct points in a vertical strip $\{z \in \mathbb{C} : -\alpha < \text{Re}(z) < 0\}$ do not satisfying (2) there exists, by Theorem 13, a dissipative operator $V = iA$ with eigenvalues $\{\mu_n\}_{n=1}^{\infty}$ and eigenvectors $\{v_n\}_{n=1}^{\infty}$, which are dense but not uniformly minimal in H . Moreover, since $\{\text{Re}(\mu_n)\}_{n=1}^{\infty}$ is a bounded sequence, by the Lumer-Phillips theorem, V generates a contractive C_0 -group. This situation contrasts with the case considered in our Theorem 12, since we construct non-dissipative infinitesimal operators with pure imaginary eigenvalues do not satisfying (2) and corresponding complete minimal system of eigenvectors, which generate unbounded C_0 -groups.

- The arguments above together with Theorem 12 show that a Theorem 1 cannot be improved.
- The construction of unbounded generator of the C_0 -group on H with non-bounded non-Riesz basis family of eigenvectors is trivial. From the other hand, the existence of unbounded generator of the C_0 -group on H with bounded non-Riesz basis family of eigenvectors is unknown.

From the proof of Theorem 12 we have the following consequence.

Corollary 14. *Let $\omega_{0,k}$ is the growth bound of the C_0 -group $e^{A_k t}$ constructed in Theorem 12 and $s(A_k) = \sup\{\text{Re}\lambda : \lambda \in \sigma(A_k)\}$ is a spectral bound of its generator A_k . Then, for any $k \in \mathbb{N}$,*

$$\omega_{0,k} = s(A_k) = 0.$$

PROOF.

$$0 \leq \omega_{0,k} = \lim_{t \rightarrow \infty} \frac{\ln \|e^{A_k t}\|}{t} \leq \lim_{t \rightarrow \infty} \frac{\ln \sqrt{4^k \frac{M}{m} \tilde{D}_k |t|^2}}{t} = 0 = s(A_k), \quad k \in \mathbb{N}.$$

□

Combining Theorem 12 and Corollary 14 with the spectral theorem of K. Boyadzhiev and R. DeLaubenfels [10] we obtain the following result.

Proposition 15. *For each $k \in \mathbb{N}$ and arbitrary $\alpha > 0$ the operator A_k constructed in Theorem 12 has a bounded \mathcal{H}^∞ -calculus on a strip $H_\alpha = \{z \in \mathbb{C} : |Re(z)| < \alpha\}$.*

It is interesting to compare this result with the construction of an operator A on H without a bounded \mathcal{H}^∞ -calculus from [12](Section 5.5). This construction is based on the concept of non-Riesz basis. About \mathcal{H}^∞ -calculus see, e.g., [10, 11, 12, 2].

Note that Theorem 8 is a special case of Theorem 12 when $k = 1$ and $f_1(x) = \ln x$. Within the context of Theorem 12 we note the following.

Remark 2.

- For each $k \in \mathbb{N}$ the operator A_k from Theorem 12 is an unbounded linear operator with $\overline{D(A_k)} = H_k(\{e_n\})$ and A_k is closed on $D(A_k)$.
- A_k has pure point spectrum $\overline{\{\lambda_n = if_k(n)\}_{n=1}^\infty}$, which does not satisfy the condition (2), and, moreover, cannot be decomposed into K sets, with every set having a uniform gap, since $f_k \in \mathcal{S}_k$.
- If we refuse the condition (2), then the converse, in some sense, statement to the Theorem 1 holds true. More precisely, suppose that $f : [1, +\infty) \mapsto \mathbb{R}$ is any real function such that $\{f(n)\}_{n=1}^\infty$ does not satisfy the condition (2) and no two points $\lambda, \mu \in \overline{\{f(n), n \in \mathbb{N}\}}$ can be joined by a segment lying entirely in $\overline{\{f(n), n \in \mathbb{N}\}}$. Now if we define $A : H \supset D(A) \mapsto H$ as $A \sum_{n=1}^\infty \alpha_n e_n = \sum_{n=1}^\infty if(n) \cdot \alpha_n e_n$, with domain

$$D(A) = \left\{ x = \sum_{n=1}^\infty \alpha_n e_n \in H : \{f(n) \cdot \alpha_n\}_{n=1}^\infty \in \ell_2 \right\},$$

then A is a Riesz-spectral operator and it generates a C_0 -group on H . For details see [23].

4. The construction of infinitesimal operators with non-basis family of eigenvectors on certain Banach spaces

The construction of infinitesimal operators with complete minimal non-basis family of eigenvectors on Banach spaces is similar to the construction of infinitesimal operators with complete

minimal non-basis family of eigenvectors on Hilbert spaces. Namely, we have the following theorem, analogous to Theorem 12.

Theorem 16. *Assume that $\{e_n\}_{n=1}^\infty$ is a symmetric basis of ℓ_p , $p > 1$, $k \in \mathbb{N}$ and let T be the right shift operator associated to $\{e_n\}_{n=1}^\infty$. Then $\{e_n\}_{n=1}^\infty$ does not form a basis of $\ell_{p,k}(\{e_n\})$ and the operator $A_k : \ell_{p,k}(\{e_n\}) \supset D(A_k) \mapsto \ell_{p,k}(\{e_n\})$, defined by*

$$A_k x = A_k(\mathfrak{f}) \sum_{n=1}^{\infty} c_{k,n} e_n = (\mathfrak{f}) \sum_{n=1}^{\infty} i f_k(n) \cdot c_{k,n} e_n,$$

where $f_k \in \mathcal{S}_k$, with domain

$$D(A_k) = \left\{ x = (\mathfrak{f}) \sum_{n=1}^{\infty} c_{k,n} e_n \in \ell_{p,k}(\{e_n\}) : \{f_k(n) \cdot c_{k,n}\}_{n=1}^{\infty} \in \ell_p(\Delta^k) \right\}, \quad (10)$$

generates a C_0 -group on $\ell_{p,k}(\{e_n\})$.

PROOF. First we note that Proposition 5 yields that $\{e_n\}_{n=1}^\infty$ does not form a basis of $\ell_{p,k}(\{e_n\})$. Since $A_k e_n = i f_k(n) \cdot e_n$, $n \in \mathbb{N}$, for every $t \in \mathbb{R}$ we have $e^{A_k t} e_n = e^{i t f_k(n)} e_n$, $n \in \mathbb{N}$, and $e^{A_k t} x = e^{A_k t}(\mathfrak{f}) \sum_{n=1}^{\infty} c_{k,n} e_n = (\mathfrak{f}) \sum_{n=1}^{\infty} e^{i t f_k(n)} c_{k,n} e_n$ for $x \in \ell_{p,k}(\{e_n\})$. The group property for $e^{A_k t}$ is obvious.

Second, we fix $k \in \mathbb{N}$ and show that $e^{A_k t} \in [\ell_{p,k}(\{e_n\})]$ for arbitrary $t \in \mathbb{R}$. To this end we note that for each $x = (\mathfrak{f}) \sum_{n=1}^{\infty} c_{k,n} e_n \in \ell_{p,k}(\{e_n\})$,

$$\begin{aligned} \|e^{A_k t} x\|_k &= \left\| (\mathfrak{f}) \sum_{n=1}^{\infty} e^{i t f_k(n)} c_{k,n} e_n \right\|_k \\ &= \left\| \sum_{n=1}^{\infty} \left(e^{i t f_k(n)} c_{k,n} - C_k^1 e^{i t f_k(n-1)} c_{k,n-1} + \dots + (-1)^k e^{i t f_k(n-k)} c_{k,n-k} \right) e_n \right\|, \end{aligned}$$

where we set $c_{k,1-j} = f_k(1-j) = 0$ for $j \in \mathbb{N}$. Further, as in the proof of Theorem 12,

$$\begin{aligned} \|e^{A_k t} x\|_k &= \left\| \sum_{n=1}^{\infty} e^{i t f_k(n)} (c_{k,n} - C_k^1 c_{k,n-1} + \dots + (-1)^k c_{k,n-k}) e_n \right. \\ &\quad + \sum_{n=2}^{\infty} C_k^1 c_{k,n-1} (e^{i t f_k(n)} - e^{i t f_k(n-1)}) e_n - \sum_{n=3}^{\infty} C_k^2 c_{k,n-2} (e^{i t f_k(n)} - e^{i t f_k(n-2)}) e_n + \dots \\ &\quad \left. + (-1)^{k+1} \sum_{n=k+1}^{\infty} c_{k,n-k} (e^{i t f_k(n)} - e^{i t f_k(n-k)}) e_n \right\|. \end{aligned}$$

Denote $\beta_{k,j}(n) = n^k(f_k(n-j) - f_k(n))$, $j = 1, \dots, k$. Observe that, since $f_k \in \mathcal{S}_k$, for any $j = 1, \dots, k$, we have that $\lim_{n \rightarrow \infty} \beta_{k,j}(n) < \infty$. Moreover, for any $j = 1, \dots, k$, and each $n \geq j+1$, we have, as in the proof of Theorem 12, that

$$e^{itf_k(n)} - e^{itf_k(n-j)} = e^{itf_k(n)} \left(1 - e^{it(f_k(n-j) - f_k(n))}\right),$$

$$1 - e^{it(f_k(n-j) - f_k(n))} \sim it(f_k(n-j) - f_k(n)) = itn^{-k}\beta_{k,j}(n),$$

for sufficiently large n . Therefore, we infer that for any $j = 1, \dots, k$, an estimate

$$\left\| \sum_{n=j+1}^{\infty} c_{k,n-j} (e^{itf_k(n)} - e^{itf_k(n-j)}) e_n \right\| \leq B_{k,j} \|x\|_k$$

takes place if and only if an estimate $\left\| \sum_{n=j+1}^{\infty} e^{itf_k(n)} c_{k,n-j} \frac{it}{n^k} \beta_{k,j}(n) e_n \right\| \leq \tilde{B}_{k,j} \|x\|_k$ holds. Since $\{e_n\}_{n=1}^{\infty}$ is a symmetric basis of ℓ_p , by virtue of Proposition 4, application of the Hardy inequality k times yields for every $j = 1, \dots, k$, the following.

$$\begin{aligned} & \left\| \sum_{n=j+1}^{\infty} e^{itf_k(n)} c_{k,n-j} \frac{it}{n^k} \beta_{k,j}(n) e_n \right\|^p = |t|^p \left\| \sum_{n=j+1}^{\infty} \left(\sum_{\nu=1}^{n-j} (c_{k,\nu} - c_{k,\nu-1}) \frac{e^{itf_k(n)}}{n^k} \beta_{k,j}(n) \right) e_n \right\|^p \\ & \leq M |t|^p \sum_{n=j+1}^{\infty} \left| \sum_{\nu=1}^{n-j} \frac{c_{k,\nu} - c_{k,\nu-1}}{n} \right|^p \left| \frac{e^{itf_k(n)}}{n^{k-1}} \beta_{k,j}(n) \right|^p < MD_{k,j} |t|^p \sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{\nu=1}^n \frac{|c_{k,\nu} - c_{k,\nu-1}|}{n^{k-1}} \right)^p \\ & \leq \alpha_p MD_{k,j} |t|^p \sum_{n=1}^{\infty} \frac{|c_{k,n} - c_{k,n-1}|^p}{n^{p(k-1)}} \\ & = \alpha_p MD_{k,j} |t|^p \sum_{n=1}^{\infty} \left(\frac{1}{n} \left| \sum_{\nu=1}^n (c_{k,\nu} - 2c_{k,\nu-1} + c_{k,\nu-2}) \right| \frac{1}{n^{k-2}} \right)^p \\ & \leq \alpha_p^2 MD_{k,j} |t|^p \sum_{n=1}^{\infty} \frac{|c_{k,n} - 2c_{k,n-1} + c_{k,n-2}|^p}{n^{p(k-2)}} \leq \dots \\ & \leq \alpha_p^k MD_{k,j} |t|^p \sum_{n=1}^{\infty} |c_{k,n} - C_k^1 c_{k,n-1} + C_k^2 c_{k,n-2} + \dots + (-1)^{k+1} C_k^{k-1} c_{k,n-k+1} + (-1)^k c_{k,n-k}|^p \\ & \leq \alpha_p^k \frac{M}{m} D_{k,j} |t|^p \|x\|_k^p, \end{aligned}$$

where $\alpha_p = \left(\frac{p}{p-1}\right)^p$. This implies that $e^{A_k t} \in [\ell_{p,k}(\{e_n\})]$ for any $t \in \mathbb{R}$.

Finally, since the proof of the strong continuity of $e^{A_k t}$ is similar to the proof of the strong continuity of $e^{A_k t}$ in Theorem 12, the Theorem 16 is proved. \square

Now define the operator $B : \ell_{p,1}(\{e_n\}) \supset D(B) \mapsto \ell_{p,1}(\{e_n\})$ as

$$Bx = B(\mathfrak{f}) \sum_{n=1}^{\infty} c_n e_n = (\mathfrak{f}) \sum_{n=1}^{\infty} \lambda_n c_n e_n, \quad (11)$$

with domain

$$D(B) = \left\{ x = (\mathfrak{f}) \sum_{n=1}^{\infty} c_n e_n \in \ell_{p,1}(\{e_n\}) : \{\lambda_n c_n\}_{n=1}^{\infty} \in \ell_p(\Delta) \right\}, \quad (12)$$

In particular case when $k = 1$ and $f_1(x) = \ln x$ we have from Theorem 16 the following immediate consequence, which is similar to Theorem 8.

Corollary 17. *Let $\{e_n\}_{n=1}^{\infty}$ be a symmetric basis of ℓ_p , $p > 1$. Then $\{e_n\}_{n=1}^{\infty}$ does not form a basis of $\ell_{p,1}(\{e_n\})$ and the operator B defined by (11) with domain (12), where $\lambda_n = i \ln n$, generates a C_0 -group on $\ell_{p,1}(\{e_n\})$.*

As in the case of Proposition 9 we note that even if we consider the spectrum $\{\lambda_n\}_{n=1}^{\infty}$ of B defined by (11,12) of the same geometric nature, i.e. satisfying (7), then B not necessary generates a C_0 -group on $\ell_{p,1}(\{e_n\})$. To show this we choose $\lambda_n = i n^{\frac{1}{p}}$ and prove the following.

Proposition 18. *Let $\{e_n\}_{n=1}^{\infty}$ be a symmetric basis of ℓ_p , $p \geq 1$. Then the operator B defined by (11) with domain (12), where $\lambda_n = i n^{\frac{1}{p}}$, does not generate a C_0 -group on $\ell_{p,1}(\{e_n\})$.*

PROOF. Observe that $e^{Bt} \in [\ell_{p,1}(\{e_n\})]$ for any $t \in \mathbb{R}$ if and only if

$$\sum_{n=2}^{\infty} (e^{itn^{\frac{1}{p}}} - e^{it(n-1)^{\frac{1}{p}}}) c_{n-1} e_n$$

converges in ℓ_p for all $x = (\mathfrak{f}) \sum_{n=1}^{\infty} c_n e_n \in D(B)$ and for any t . Since for $n \geq 2$,

$$e^{itn^{\frac{1}{p}}} - e^{it(n-1)^{\frac{1}{p}}} = e^{itn^{\frac{1}{p}}} \left(1 - e^{it((n-1)^{\frac{1}{p}} - n^{\frac{1}{p}})} \right),$$

and using the asymptotics

$$1 - e^{it((n-1)^{\frac{1}{p}} - n^{\frac{1}{p}})} \sim it \left((n-1)^{\frac{1}{p}} - n^{\frac{1}{p}} \right) \sim -itn^{\frac{1-p}{p}}$$

for large n , we arrive at the asymptotic series

$$it \sum_{n=2}^{\infty} e^{itn^{\frac{1}{p}}} n^{\frac{1-p}{p}} c_{n-1} e_n.$$

Further, by Proposition 4, we have the following estimate

$$\begin{aligned} \left\| it \sum_{n=2}^{\infty} e^{itn^{\frac{1}{p}}} n^{\frac{1-p}{p}} c_{n-1} e_n \right\|^p &= |t|^p \left\| \sum_{n=2}^{\infty} \left(\sum_{j=1}^{n-1} (c_j - c_{j-1}) e^{itn^{\frac{1}{p}}} n^{\frac{1-p}{p}} \right) e_n \right\|^p \\ &\geq m|t|^p \sum_{n=2}^{\infty} \left| \sum_{j=1}^{n-1} \frac{c_j - c_{j-1}}{n^{1-\frac{1}{p}}} \right|^p \left| e^{itn^{\frac{1}{p}}} \right|^p = m|t|^p \sum_{n=1}^{\infty} \left| \sum_{j=1}^n \frac{c_j - c_{j-1}}{(n+1)^{1-\frac{1}{p}}} \right|^p. \end{aligned}$$

Combining this estimate together with the fact that the operator

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ 2^{\frac{1}{p}-1} & 2^{\frac{1}{p}-1} & 0 & 0 & \dots \\ 3^{\frac{1}{p}-1} & 3^{\frac{1}{p}-1} & 3^{\frac{1}{p}-1} & 0 & \dots \\ 4^{\frac{1}{p}-1} & 4^{\frac{1}{p}-1} & 4^{\frac{1}{p}-1} & 4^{\frac{1}{p}-1} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

is unbounded in ℓ_p , we conclude that there exists $t \neq 0$ such that $e^{Bt} \notin [\ell_{p,1}(\{e_n\})]$. Therefore B does not generate a C_0 -group on $\ell_{p,1}(\{e_n\})$. \square

Using the same arguments as in the proof of Proposition 18, we can say more.

Proposition 19. *Let $\{e_n\}_{n=1}^{\infty}$ be a symmetric basis of ℓ_p , $p \geq 1$. Then the operator B defined by (11) with domain (12), where $\{\lambda_n\}_{n=1}^{\infty}$ satisfy (7) and $\lim_{n \rightarrow \infty} n^{-\frac{1}{p}} |\lambda_n| > 0$, does not generate a C_0 -group on $\ell_{p,1}(\{e_n\})$.*

From the proof of Theorem 16 we have a consequence, similar to the Corollary 14.

Corollary 20. *Let $\omega_{0,k}$ is the growth bound of the C_0 -group $e^{A_k t}$ constructed in Theorem 16 and $s(A_k) = \sup\{\operatorname{Re} \lambda : \lambda \in \sigma(A_k)\}$ is a spectral bound of its generator A_k . Then, for any $k \in \mathbb{N}$,*

$$\omega_{0,k} = s(A_k) = 0.$$

PROOF.

$$0 \leq \omega_0 = \lim_{t \rightarrow \infty} \frac{\ln \|e^{A_k t}\|}{t} \leq \lim_{t \rightarrow \infty} \frac{\ln \left(\alpha_p^k \frac{M}{m} \tilde{D}_k |t|^p \right)^{\frac{1}{p}}}{t} = 0 = s(A_k), \quad k \in \mathbb{N}.$$

\square

Within the context of Theorem 16 we also note the following.

Remark 3.

- For each $k \in \mathbb{N}$ the operator A_k from Theorem 16 is an unbounded linear operator with $\overline{D(A_k)} = \ell_{p,k}(\{e_n\})$ and A_k is closed on $D(A_k)$.
- A_k has pure point spectrum $\overline{\{\lambda_n = if_k(n)\}_{n=1}^\infty}$, which does not satisfy the condition (2), and, moreover, cannot be decomposed into K sets, with every set having a uniform gap, since $f_k \in \mathcal{S}_k$.
- Suppose that $f : [1, +\infty) \mapsto \mathbb{R}$ is any real function such that no two points $\lambda, \mu \in \overline{\{f(n), n \in \mathbb{N}\}}$ can be joined by a segment lying entirely in $\overline{\{f(n), n \in \mathbb{N}\}}$. If we define $A : \ell_p \supset D(A) \mapsto \ell_p$ as $A \sum_{n=1}^\infty \alpha_n e_n = \sum_{n=1}^\infty if(n) \cdot \alpha_n e_n$, with domain

$$D(A) = \left\{ x = \sum_{n=1}^\infty \alpha_n e_n \in \ell_p : \{f(n) \cdot \alpha_n\}_{n=1}^\infty \in \ell_p \right\},$$

then it can be shown that A generates a C_0 -group on ℓ_p .

5. Concluding remarks

The results of the present paper allow us to say the following. A Theorem 1 cannot be improved. Moreover, it is impossible to obtain any analog of Theorem 1 concerning non-basis family of eigenvectors by means of refusing or weakening of the condition (2). On the other hand, it is interesting to obtain some analogs of Theorem 1 in Banach spaces with certain classes of bases, e.g., symmetric bases, unconditional bases.

Theorem 16 and Remark 3 allow us to say that symmetric bases in ℓ_p spaces behave like Riesz bases in H . Consequently, we arrive at the idea of possible generalization of the Theorem 1 to the case of operators with symmetric basis family of eigenvectors, which generate C_0 -groups on the spaces ℓ_p , $p > 1$, and propose the following.

Conjecture 21. *Let A be the generator of the C_0 -group on the space ℓ_p , $p > 1$, with eigenvalues $\{\lambda_n\}_{n=1}^\infty$ (counting with multiplicity) and the corresponding (normalized) eigenvectors $\{e_n\}_{n=1}^\infty$. If $\overline{\text{Lin}\{e_n\}_{n=1}^\infty} = \ell_p$ and the point spectrum $\{\lambda_n\}_{n=1}^\infty$ satisfies (2), then $\{e_n\}_{n=1}^\infty$ forms a symmetric basis of ℓ_p .*

Finally, the answer on the following important question does not yet exist. Is it possible to construct the unbounded generator of a C_0 -group with bounded non-Riesz basis family of eigenvectors? Note that the construction of unbounded generator of the C_0 -semigroup with bounded non-Riesz basis family of eigenvectors is trivial, see, e.g., [12](Section 5.5).

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